KANT'S PHILOSOPHY OF MATHEMATICS

By Dr. Marsigit, M.A.
Yogyakarta State University, Yogyakarta, Indonesia
Email: marsigitina@yahoo.com, Web: http://powermathematics.blogspot.com
HomePhone: 62 274 886 381; MobilePhone: 62 815 7870 8917

Kant's [1] *philosophy of mathematics* plays a crucial role in his *critical philosophy*, and a clear understanding of his notion of mathematical construction would do much to elucidate his general epistemology. Friedman M. in Shabel L. insists that Kant's philosophical achievement consists precisely in the depth and acuity of his insight into the state of the mathematical exact sciences as he found them, and, although these sciences have radically changed in ways, this circumstance in no way diminishes Kant's achievements. Friedman M [2] further indicates that the highly motivation to uncover Kant's *philosophy of mathematics* comes from the fact that Kant was deeply immersed in the textbook mathematics of the eighteenth century. Since Kant's *philosophy of mathematics*[3] was developed relative to a specific body of mathematical practice quite distinct from that which currently obtains, our reading of Kant must not ignore the dissonance between the ontology and methodology of *eighteenth*- and *twentieth-century* mathematics. The description of Kant's philosophy

[1] Shabel, L., 1998, " *Kant on the 'Symbolic Construction' of Mathematical Concepts",* Pergamon Studies in History and Philosophy of Science , Vol. 29, No. 4, p. 592
[2] In Shabel, L., 1998, " *Kant on the 'Symbolic Construction' of Mathematical Concepts",* Pergamon Studies in History and Philosophy of Science , Vol. 29, No. 4, p. 595
[3] Shabel, L., 1998, " *Kant on the 'Symbolic Construction' of Mathematical Concepts",* Pergamon Studies in History and Philosophy of Science , Vol. 29, No. 4, p. 617

of mathematics involves the discussion of Kant's perception on the basis validity of mathematical knowledge which consists of arithmetical knowledge and geometrical knowledge. It also needs to elaborate Kant perception on mathematical judgment and on the construction of mathematical concepts and cognition as well as on mathematical method.

Some writers may perceive that Kant's philosophy of mathematics consists of *philosophy of geometry*, bridging from his theory of *space* to his doctrine of *transcendental idealism*, which is parallel with the *philosophy of arithmetic* and *algebra*. However, it was suggested that Kant's *philosophy of mathematics* would account for the construction in intuition of all *mathematical concepts*, not just the obviously constructible concepts of *Euclidean geometry*. Attention to his back ground will provide facilitates a strong reading of Kant's *philosophy of mathematics* which is historically accurate and well motivated by Kant's own text. The argument from *geometry* exemplifies *a synthetic argument* that reasons progressively from a theory of *space* as *pure intuition.* Palmquist S.P. (2004) denotes that in the light of Kant's *philosophy of mathematics*, there is a new trend in the *philosophy of mathematics* i.e. *the trend away from any attempt to give definitive statements as to what mathematics is.*

A. Kant on the Basis Validity of Mathematical Knowledge

According to Wilder R.L., Kant's *philosophy of mathematics* can be interpreted in a *constructivist manner* and *constructivist ideas* that presented in the nineteenth century-notably by Leopold Kronecker, who was an important for a runner of *intuitionism*-in opposition to the tendency in mathematics toward *set-theoretic* ideas, long before the *paradoxes of set theory* were discovered. In his *philosophy of mathematics*[4], Kant supposed that *arithmetic* and *geometry* comprise *synthetic a priori judgments* and that *natural science* depends on them for its power to explain and predict events. As *synthetic a priori judgments*[5], the *truths of mathematics* are both *informative* and *necessary;* and since mathematics derives from our own *sensible intuition,* we can be absolutely sure that it must apply to everything we perceive, but for the same reason we can have no assurance that it has anything to do with the way things are apart from our perception of them.

Kant[6] believes that *synthetic a priori propositions* include both *geometric propositions* arising from innate spatial geometric intuitions and *arithmetic propositions* arising from innate intuitions about *time* and *number*. The belief in *innate intuitions* about *space* was discredited by the discovery of *non-Euclidean geometry,*

[4] Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p.205
[5] *Ibid.205*
[6] Wegner, P., 2004, "*Modeling, Formalization, and Intuition."* Department of Computer Science. Retrieved 2004 <*http://www.google.com/ wiki/Main+Page*>

which showed that alternative geometries were consistent with physical reality. Kant[7] perceives that *mathematics* is about the *empirical world*, but it is special in one important way. *Necessary properties* of the world are found through *mathematical proofs*. To prove something is wrong, one must show only that the world could be different. While[8], *sciences* are basically *generalizations from experience*, but this can provide only *contingent* and *possible properties* of the world. *Science* simply predicts that the *future* will mirror the *past.*

In his *Critic of Pure Reason* Kant defines *mathematics* as *an operation of reason by means of the construction of conceptions to determine a priori an intuition in space (its figure), to divide time into periods, or merely to cognize the quantity of an intuition in space and time, and to determine it by number. Mathematical rules*[9], current in the field of common experience, and which common sense stamps everywhere with its approval, are regarded by them as *mathematical axiomatic*. According to Kant[10], the *march of mathematics* is pursued from the validity from what source the conceptions of *space* and *time* to be examined into the origin of the pure conceptions of the understanding. The essential and distinguishing feature[11] of *pure mathematical cognition* among all other *a priori cognitions* is, that it cannot at all proceed from concepts, but only by means of the *construction of concepts.*

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[7] Posy, C. ,1992, "*Philosophy of Mathematics*", Retreived 2004 <*http://www.cs.washington.edu/ homes/ gjb.doc/philmath.htm*>
[8] Ibid.
[9] Kant, I., 1781, "*The Critic Of Pure Reason: SECTION III. Of Opinion, Knowledge, and Belief; CHAPTER III. The Arehitectonic of Pure Reason*" Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>
[10] *Ibid.*
[11] Kant, I, 1783, *Prolegomena To Any Future Methaphysics,* Preamble, p. 19

Kant[12] conveys that *mathematical judgment* must proceed beyond the concept to that which its corresponding visualization contains. *Mathematical judgments* neither can, nor ought to, arise *analytically*, by dissecting the concept, but are all *synthetical*. From the observation on *the nature of mathematics*, Kant[13] insists that *some pure intuition* must form mathematical basis, in which all its concepts can be exhibited or constructed, *in concreto* and yet *a priori*. Kant[14] concludes that *synthetical propositions a priori are possible in pure mathematics, if we can locate this pure intuition and its possibility*. The intuitions[15] which pure mathematics lays at the foundation of all its cognitions and judgments which appear at once *apodictic* and *necessary* are *Space* and *Time*. For mathematics[16] must first have all its concepts in intuition, and pure mathematics in pure intuition, it must construct them. Mathematics[17] proceeds, not *analytically* by dissection of concepts, but *synthetically;* however, if pure intuition be wanting, it is impossible for *synthetical judgments a priori* in mathematics.

The basis of mathematics[18] *actually are pure intuitions*, which make its *synthetical* and *apodictically valid propositions possible. Pure Mathematics*, and especially *pure geometry*, can only have *objective reality* on condition that they refer

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[12] *Ibid. p. 21*

[13] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* First Part Sect. 7", Trans. Paul Carus. Retrieved 2003 <*www. phil-books.com/* >

[14] *Ibid.*

[15] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* First Part Sect.10", Trans. Paul Carus. Retrieved 2003 <*www. phil-books.com/* >

[16] *Ibid.*

[17] *Ibid.*

[18] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* First Part Sect.12 Trans. Paul Carus. Retrieved 2003 <*www. phil-books.com/* >

to objects of sense. The *propositions of geometry*[19] are not the results of a mere creation of our *poetic imagination*, and that therefore they cannot be referred with assurance to actual objects; but rather that they are *necessarily valid of space*, and consequently of all that may be found in *space*, because *space* is nothing else than the *form of all external appearances,* and it is this form alone where objects of sense can be given. The *space*[20] of the geometer is exactly the *form of sensuous intuition* which we find *a priori* in us, and contains the *ground of the possibility of all external appearances.* In this way[21] *geometry* be made secure, for objective reality of its propositions, from the intrigues of a shallow metaphysics of the un-traced sources of their concepts.

Kant[22] argues that *mathematics is a pure product of reason, and moreover is thoroughly synthetical.* Next, the question arises: Does not this faculty, which produces mathematics, as it neither is nor can be based upon experience, presuppose some ground of cognition *a priori*,[23] which lies deeply hidden, but which might reveal itself by these its effects, if their first beginnings were but diligently *ferreted out*? However, Kant[24] found that *all mathematical cognition has this peculiarity: it must first exhibit its concept in a visual intuition and indeed a priori, therefore in an*

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[19] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* REMARK 1 Trans. Paul Carus. Retrieved 2003 <*www. phil-books.com/* >

[20] *Ibid.*

[21] *Ibid.*

[22] Wikipedia The Free Encyclopedia. Retrieved 2004 <*http:/en.wikipedia.org/*>

[23] *Ibid.*

[24] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* First Part Of The Transcendental Problem: How Is Pure Mathematics Possible? Sect. 6. p. 32

intuition which is not empirical, but pure. Without this[25] mathematics cannot take a single step; hence its judgments are always visual, viz., *intuitive*; whereas philosophy must be satisfied with *discursive judgments* from mere concepts, and though it may illustrate its doctrines through a visual figure, can never derive them from it.

1. The Basis Validity of the Concept of Arithmetic

In his *Critic of Pure Reason* Kant reveals that *arithmetical propositions* are *synthetical*. To show this, Kant[26] convinces it by trying to get a large numbers of evidence that without having recourse to intuition or mere analysis of our conceptions, it is impossible to arrive at the sum total or product. In *arithmetic*[27], intuition must therefore here lend its aid only by means of which our synthesis is possible. *Arithmetical judgments*[28] are therefore *synthetical* in which we can analyze *our concepts* without calling *visual images* to our aid as well as we can never find the arithmetical sum by such mere dissection.

[25] Immanuel Kant, *Prolegomena to Any Future Metaphysics*, First Part Of The Transcendental Problem: How Is Pure Mathematics Possible? Sect. 7.p. 32
[26] Kant, I., 1787, "*The Critic Of Pure Reason: INTRODUCTION: V. In all Theoretical Sciences of Reason, Synthetical Judgements "a priori" are contained as Principles"* Translated By J. M. D. Meiklejohn, Retrieved 2003 <*http://Www.Encarta.Msn. Com/*>)
[27] *Ibid.*
[28] Kant, I, 1783. *"Prolegomena to Any Future Metaphysic: Preamble On The Peculiarities Of All Metaphysical Cognition, Sec.2"* Trans. Paul Carus.. Retrieved 2003 <*www. phil-books.com/* >

Kant[29] propounds that *arithmetic* accomplishes its *concept of number* by the *successive addition* of units in *time*; and *pure mechanics* especially cannot attain its concepts of motion without employing the representation of *time*. Both *representations*[30], however, are only intuitions because if we omit from the empirical intuitions of bodies and their alterations everything empirical or belonging to sensation, *space* and *time* still remain. According to Kant[31], *arithmetic* produces its concepts of number through *successive addition* of units in *time*, and pure mechanics especially can produce its concepts of motion only by means of the representation of *time*. Kant[32] defines the *schema of number* in exclusive reference to time; and, as we have noted, it is to this definition that Schulze appeals in support of his view of arithmetic as the science of counting and therefore of *time*. It at least shows that Kant perceives some form of connection to exist between arithmetic and *time*.

Kant[33] is aware that *arithmetic* is related closely to the *pure categories* and to *logic*. A fully explicit awareness of number goes the successive apprehension of the stages in its construction, so that the structure involved is also represented by a sequence of *moments of time*. *Time*[34] thus provides a realization for any number which can be realized in experience at all. Although this view is plausible enough, it does not seem strictly necessary to preserve the connection with *time* in the *necessary*

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[29] Kant, I, 1783. "*Prolegomena to Any Future Metaphysic: First Part Of The Transcendental Problem: How Is Pure Mathematics Possible?"* Trans. Paul Carus.. Retrieved 2003 <*www. phil-books.com/* >
[30] *Ibid.*
[31] Smith, N. K., 2003, *"A Commentary to Kant's Critique of Pure Reason: Kant on Arithmetic,",* New York: Palgrave Macmillan. *p. 128*
[32] *Ibid. p. 129*
[33] *Ibid. p. 130*
[34] *Ibid. p. 131*

extrapolation beyond actual experience. Kant[35], as it happens, did not see that arithmetic could be *analytic*. He explained the following:

> Take an example of *"7 + 5 = 12"* . If *"7 + 5"* is understood as the subject, and "12" as the predicate, then the concept or meaning of "12" does not occur in the subject; however, intuitively certain that *"7 + 5 = 12"* cannot be denied without contradiction. In term of the development of propositional logic, proposition like "P or not P" clearly cannot be denied without contradiction, but it is not in a subject-predicate form. Still, *"P or not P*" is still clearly about two identical things, the P's, and *"7 + 5 = 12"* is more complicated than this. But, if *"7 + 5 = 12*" could be derived directly from logic, without substantive axioms like in geometry, then its *analytic* nature would be certain.

Hence[36], thinking of arithmetical construction as a process in *time* is a useful picture for interpreting problems of the mathematical constructivity. Kant argues[37] that in order to verify *"7+5=12"*, we must consider an instance.

2. The Basis Validity of the Concept of Geometrical

In his *Critic of Pure Reason* (1787) Kant elaborates that *geometry* is based upon the *pure intuition of space*; and, *arithmetic* accomplishes its concept of number by the *successive addition* of units in *time*; and pure mechanics especially cannot attain its concepts of motion without employing the *representation of time*. Kant[38] stresses that both *representations*, however, are only *intuitions*; for if we omit from the empirical intuitions of bodies and their alterations (motion) everything *empirical*, or

[35] Ross, K.L., 2002, "*Immanuel Kant (1724-1804)"* Retreived 2003 <*http://www. Friesian.com/ross/*>
[36] *Ibid.*
[37] Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p. 198
[38] Kant, I, 1783. *"Prolegomena to Any Future Metaphysic:* , First Part Of The Transcendental Problem: How Is Pure Mathematics Possible? Sect.10, p. 34

belonging to sensation, *space* and *time* still remain. Therefore, Kant[39] concludes that *pure mathematics* is *synthetical cognition a priori*. *Pure mathematics* is only *possible* by referring to no other objects than those of the senses, in which, at the basis of their empirical intuition lies a pure intuition of *space* and *time* which is *a priori*.

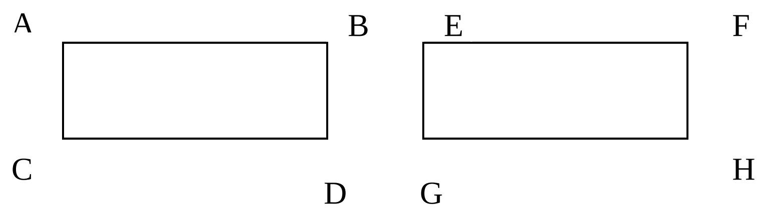

Figure 14: Proof of the complete congruence
of two given figures

Kant[40] illustrates, see Figure 14, that in *ordinary* and *necessary* procedure of geometers, all proofs of the complete congruence of two given figures come ultimately to to coincide; which is evidently nothing else than a *synthetical* proposition resting upon *immediate intuition*. This intuition must be *pure* or given *a priori*, otherwise the proposition could not rank as *apodictically certain*, but would have *empirical certainty* only. Kant[41] further claims that everywhere *space* has *three dimensions* (Figure15).

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[39] *Ibid. p. 35*
[40] Kant, I., 1787, *"The Critic Of Pure Reason: SS 9 General Remarks on Transcendental Aesthetic."* Translated By J. M. D. Meiklejohn, Retrieved 2003 <*http://Www.Encarta.Msn. Com/*>
[41] *Ibid.*

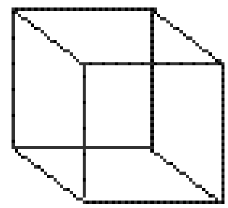

Figure 15: Three dimensions space

This claim is based on the proposition that *not more than three lines can intersect at right angles in one point* (Figure 16).

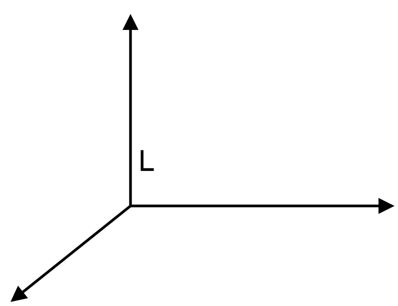

Figure 16: Three lines intersect perpendicularly at one point

Kant[42] argues that drawing the line to infinity and representing the series of changes e.g. spaces travers by motion can only attach to *intuition*, then he concludes that the *basis of mathematics* actually are *pure intuitions;* while the *transcendental deduction* of the notions of *space* and of *time* explains the *possibility of pure mathematics.*

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[42] *Ibid.*

Kant[43] defines that *geometry* is a science which determines the properties of *space synthetical*ly, and yet *a priori*. What, then, must be our representation of *space*, in order that such a cognition of it may be possible? Kant[44] explains that it must be *originally intuition*, for from a mere conception, no propositions can be deduced which go out beyond the conception, and yet this happens in geometry. But this intuition must be found in the mind *a priori*, that is, before any perception of objects, consequently must be *pure*, not empirical, *intuition.* According to Kant[45], *geometrical principles* are always *apodeictic*, that is, *united with the consciousness of their necessity*; however, propositions as "s*pace has only three dimensions*", cannot be *empirical judgments* nor conclusions from them. Kant[46] claims that it is only by means of our explanation that the possibility of geometry, as a *synthetical* science *a priori*, becomes comprehensible.

As the propositions of geometry[47] are cognized *synthetical*ly *a priori*, and with *apodeictic certainty*. According to Kant[48], *all principles of geometry* are no less *analytical*; and it based upon the *pure intuition of space.* However, the *space* of the geometer[49] would be considered a mere *fiction*, and it would not be credited with objective validity, because we cannot see how things must of necessity agree with an image of them, which we make spontaneously and previous to our acquaintance with

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[43] *Ibid.*
[44] *Ibid.*
[45] *Ibid.*
[46] *Ibid.*
[47] *Ibid.*
[48] *Ibid.*
[49] Kant, I, 1783, *"Prolegomena to Any Future Metaphysic: REMARK 1"* Trans. Paul Carus.. Retrieved 2003 <*www. phil-books.com/* >

them. But if the image[50] is the essential property of our sensibility and if this sensibility represents not things in themselves, we shall easily comprehend that all external objects of our world of sense must necessarily coincide in the most rigorous way with the propositions of geometry. The *space* of the geometer[51] is exactly the *form of sensuous intuition* which we find *a priori* and contains the ground of the possibility of all external appearances.

In his own remarks on geometry, Kant[52] regularly cites *Euclid's angle-sum theorem* as a paradigm example of a *synthetic a priori judgment* derived via the constructive procedure that he takes to be unique to mathematical reasoning.

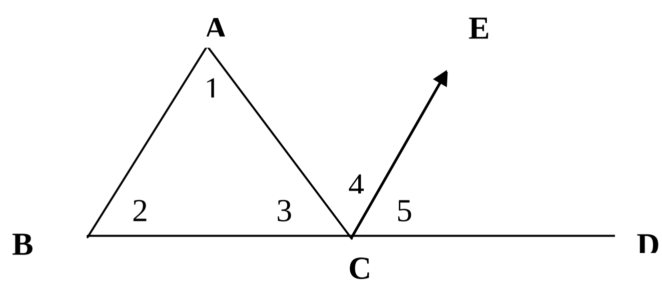

Figure 17: Euclid's angle-sum theorem

Kant describes the sort of procedure that leads the geometer to *a priori* cognition of the necessary and universal truth of the *angle-sum theorem* as (Figure 17):

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[50] *Ibid.*
[51] *Ibid.*
[52] Shabel, L., 1998, "*Kant's "Argument from Geometry*", Journal of the History of Philosophy, The Ohio State University, p.24

> The object of the theorem—the constructed triangle—is in this case "determined in accordance with the conditions of…pure intuition." The triangle is then "assessed in concreto" in pure intuition and the resulting cognition is pure and *a priori*, thus rational and properly mathematical. To illustrate, I turn to Euclid's demonstration of the angle-sum theorem, a paradigm case of what Kant considered *a priori* reasoning based on the ostensive but pure construction of mathematical concepts. Euclid reasons as follows: given a triangle ABC , extend the base BC to D. Then construct a line through C to E such that CE is parallel to AB. Since AB is parallel to CE and AC is a transversal, angle 1 is equal to angle 1'. Likewise, since BD is a transversal, angle 2 [53]

For Kant[54], the *axioms* or *principles* that ground the constructions of *Euclidean geometry* comprise the features of *space* that are cognitively accessible to us immediately and uniquely, and which precede the actual practice of geometry. Kant[55] said that *space* is *three dimensional;* two *straight lines* cannot enclose a *space*; a *triangle* cannot be constructed except on the condition that *any two of its sides are together longer than the third* (Figure 18).

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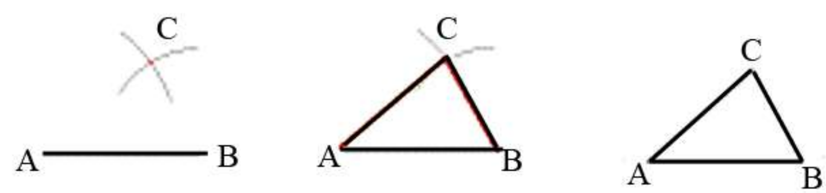

Figure 18 : Construction of triangle

[53] *Ibid. p. 28*
[54] *Ibid.p.30*
[55] *Ibid.p.30*

Kant[56] takes the procedure of describing geometrical *space* to be pure, or *a priori*, since it is performed by means of a prior pure intuition of *space* itself. According to Kant, our cognition of individual spatial regions is *a priori* since they are cognized in, or as limitations on, the *essentially single* and all encompassing *space* itself.

Of the truths of geometry[57] e.g. in performing the geometric proof on a triangle that *the sum of the angles of any triangle is 180°*, it would seem that our constructed *imaginary triangle* is operated on in such a way as to ensure complete independence from any particular empirical content. So, in term of *geometric truths*, Kant[58] might suggest that they are *necessary truths* or are they *contingent* viz. it being possible to imagine otherwise. Kant[59] argues that *geometric truth*[60] in general *relies on human intuition*, and *requires a synthetic addition* of information from our *pure intuition of space,* which is a *three-dimensional Euclidean space*. Kant does not claim that the idea of such intuition can be reduced out to make the truth *analytic*.

In the *Prolegomena*, Kant [61] gives an everyday example of a *geometric necessary truth* for humans that a left and right hand are incongruent (See Figure 19).

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[56] *Ibid.p.32*
[57] …., 1987, "*Geometry: Analytic, Synthetic A Priori, or Synthetic A Posteriori*?", Encyclopedic Dictionary of Mathematics, Vol. I., "Geometry", , The MIT Press, p. 685
[58] *Ibid. p. 686*
[59] *Ibid. p. 689*
[60] *Ibid. p.690*
[61] *Ibid*. p.691

Figure19: Left and right hand

The notion of "*hand*" here need not be understood as the empirical object *hand*. According to Kant, we can assume that our *pure intuition filter* has adequately abstracted our *hand-experience* into something detached from its empirical component, so we are merely dealing with *a three-dimensional geometric figure shaped* like a *hand*. By "*incongruent*", the geometer simply means that no matter how we move one figure around in relation to the other, we cannot get the two figures to coincide, to match up perfectly. Kant points[62] out, there is still something *true* about the *3-D Euclidean* case that has some kind of priority over the other cases. *Synthetically*, it is *necessarily true* that the figures are *incongruent*, since the choice of view point in point of fact no choice at all.

[62] *Ibid. p.692*

B. Kant on Mathematical Judgment

In his *Critic of Pure Reason* Kant mentions that a *judgment* is the *mediate cognition of an object;* consequently it is the representation of a representation of it. In every *judgment* there is a conception which applies to his last being immediately connected with an object. All *judgments*[63] are functions of unity in our representations. A higher representation is used for our cognition of the object, and thereby many possible cognitions are collected into one. Hanna R. learns that in term of the quantity of judgments Kant captures the basic ways in which the comprehensions of the constituent concepts of a simple monadic categorical proposition are logically combined and separated.

For Kant[64], the form "*All* Fs *are* Gs*"* is *universal judgments*, the form "Some *Fs* are *Gs*" is *particular judgments*. Tthe form "This *F* is *G*" or "The *F* is *G*" is *singular judgments.* A simple monadic categorical judgment [65] can be either existentially posited or else existentially cancelled. Further, the form "it is the case that *Fs* are *Gs*" (or more simply: "*Fs* are *Gs*") is *affirmative judgment*. The form "no *Fs* are *Gs*" is *negative judgments*, and the form "*Fs* are non-*Gs*" is *infinite judgments.* Kant's pure general logic[66] includes no logic of relations or multiple quantification, because

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[63] Kant, I., 1781, "*The Critic Of Pure Reason: Transcendental Analytic, Book I, Section 1, Ss 4.",* Translated By J. M. D. Meiklejohn, Retrieved 2003 <*http://www.encarta.msn. com/*>

[64] Hanna, R., 2004, "*Kant's Theory of Judgment",* Stanford Encyclopedia of Philosophy, Retreived 2004, <*http://plato.stanford.edu/cgi-bin/encyclopedia/ archinfo.cgi?entry=kant-judgment*>

[65] *Ibid.*

[66] *Ibid.*

mathematical relations generally are represented spatiotemporally in pure or formal intuition, and *not* represented logically in the understanding. *True mathematical propositions*, for Kant[67], *are not truths of logic* viz. all *analytic* truths or *concept-based truths*, but are *synthetic truths* or *intuition-based truths.* Therefore, according to Kant[68], by the very nature of mathematical truth, there can be no such thing as an authentically "*mathematical logic.*"

For Kant[69], in term of the *relation* of judgments, *1-place subject-predicate propositions* can be either *atomic* or *molecular*; therefore, the categorical judgments repeat the simple atomic *1-place subject-predicate* form "*Fs* are *Gs*". The *molecular hypothetical judgments*[70] are of the form "If *Fs* are *Gs*, then *Hs* are *Is*" (or: "If *P* then *Q*"); and *molecular disjunctive judgments* are of the form "Either *Fs* are *Gs*, or *Hs* are *Is*" (or: "Either *P* or *Q*"). The *modality of a judgment*[71] are the basic ways in which truth can be assigned to simple *1-place subject-predicate propositions* across logically possible worlds--whether to some worlds *(possibility)*, to this world alone (*actuality*), or to all worlds (*necessity)*. Further, the *problematic judgments*[72] are of the form "Possibly, *Fs* are *Gs*" (or: "Possibly *P*"); the *ascertoric judgments* are of the form "Actually, *Fs* are *Gs*" (or: "Actually *P*"); and *apodictic judgments* are of the form "Necessarily, *Fs* are *Gs*" (or: "Necessarily *P*").

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[67] *Ibid.*
[68] *Ibid.*
[69] *Ibid*.
[70] *Ibid.*
[71] *Ibid.*
[72] *Ibid.*

Mathematical judgments [73] are all *synthetical*; and the conclusions of mathematics, as is demanded by all *apodictic certainty*, are all proceed according to *the law of contradiction.* A *synthetical proposition*[74] can indeed be comprehended according to the *law of contradiction,* but only by presupposing another *synthetical proposition* from which it follows, but never in itself. In the case of addition *7 + 5 = 12,* it[75] might at first be thought that the proposition *7 + 5 = 12 i*s a mere *analytical judgment,* following from the concept of *the sum of seven and five*, according to the law of contradiction. However, if we closely examine the operation, it appears that the concept of the sum of *7+5* contains merely their union in a single number, without its being at all thought what the particular number is that unites them.

Therefore, Kant[76] concludes that the *concept of twelve i*s by no means thought by merely thinking of the combination of seven and five; and analyzes this possible sum as we may, we shall not discover twelve in the concept. Kant[77] suggests that first of all, we must observe that *all proper mathematical judgments* are *a priori*, and *not empirical.* According to Kant[78], mathematical judgments carry with them *necessity*, which cannot be obtained from experience, therefore, it implies that it contains *pure a priori* and not *empirical cognitions.* Kant, says that we must go beyond these concepts, by calling to our aid some concrete image *[Anschauung]*, i.e., either our five fingers, or five points and we must add successively the units of the five, given in

[73] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic,* p. 15
[74] Ibid. p. 16
[75] *Ibid. p. 18*
[76] *Ibid*. p.18
[77] *Ibid. p. 19*
[78] *Ibid.p.20*

some concrete image *[Anschauung]*, to the concept of seven; hence our concept is really amplified by the proposition 7 + *5* = I 2, and we add to the first a second, not thought in it". [79] Ultimately, Kant [80] concludes that *arithmetical judgments* are therefore *synthetical*. According to Kant, we analyze our concepts without calling visual images *(Anscliauung)* to our aid. We can never find the sum by such mere dissection. Further, Kant argues that *all principles of geometry* are no less *analytical*.

Kant [81] illustrates that the proposition "*a straight line is the shortest path between two points"*, is a *synthetical proposition* because the *concept of straight* contains nothing of *quantity*, but only a *quality*. Kant then claims that the *attribute of shortness* is therefore altogether *additional*, and cannot be obtained by any analysis of the concept; and its visualization *[Anschauung]* must come to aid us; and therefore, it alone makes the synthesis possible. Kant [82] confronts the previous geometers assumption which claimed that other mathematical principles are indeed actually *analytical* and depend on *the law of contradiction*. However, he strived to show that in the case of *identical propositions*, as a *method of concatenation*, and *not as principles*, e. *g.*, "*a=a", "the whole is equal to itself", or "a + b > a"*, and *"the whole is greater than its part"*. Kant[83] then claims that although they are recognized as *valid* from mere *concepts*, they are only admitted in *mathematics*, because they can be represented in some *visual form [Anschauung]*.

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[79] *Ibid*. p.21
[80] *Ibid. p.21*
[81] Ibid p.22
[82] *Ibid. p.22*
[83] *Ibid. p.23*

C. Kant on the Construction of Mathematical Concepts and Cognition

In his *Critic of Pure Reason*, Kant ascribes that *mathematics* deals with conceptions applied to *intuition*. *Mathematics* is a *theoretical sciences* which have to determine their objects *a priori*. To demonstrate the properties of the *isosceles triangle* (Figure 20), it is not sufficient to meditate on the figure but that it is necessary to produce these properties by a *positive a priori construction.*

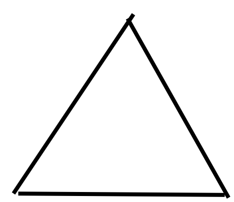

Figure 20: Isosceles triangle

According to Kant, in order to arrive with *certainty* at *a priori cognition,* we must not attribute to the object any other properties than those which necessarily followed from that which he had himself placed in the object. *Mathematician*[84] occupies himself with objects and cognitions only in so far as they can be represented by means of *intuition*; but this circumstance is easily overlooked, because the said intuition can itself be given *a priori*, and therefore is hardly to be distinguished from a mere pure conception.

[84] Kant, I., 1781, *“The Critic Of Pure Reason: Preface To The Second Edition*”, Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>

The *conception of twelve*[85] is by no means obtained by merely cogitating the *union of seven* and *five;* and we may analyze our conception of such a possible sum as long as we will, still we shall never discover in it the *notion of twelve.* Kant[86] says that we must go beyond these conceptions, and have recourse to an intuition which corresponds to one of the two-our five fingers, add the units contained in the five given in the intuition, to the conception of seven.

Further Kant states:

> For I first take the number 7, and, for the conception of 5 calling in the aid of the fingers of my hand as objects of intuition, I add the units, which I before took together to make up the number 5, gradually now by means of the material image my hand, to the number 7, and by this process, I at length see the number 12 arise. That 7 should be added to 5, I have certainly cogitated in my conception of a sum = 7 + 5, but not that this sum was equal to 12. [87]

Arithmetical propositions[88] are therefore always *synthetical*, of which we may become more clearly convinced by trying large numbers. For it[89] will thus become quite evident that it is impossible, without having recourse to intuition, to arrive at the sum total or product by means of the mere analysis of our conceptions, just as little is any principle of pure geometry *analytical*.

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[85] *Ibid.*
[86] *Ibid.*
[87] *Ibid.*
[88] *Ibid.*
[89] *Ibid.*

Figure 21: The shortest distance

In *a straight line between two points*[90], the *conception of the shortest* is therefore more wholly *an addition*, and by no analysis can it be extracted from our conception of a *straight line* (see Figure 21). Kant[91] sums up that *intuition* must therefore here lend its aid in which our synthesis is possible.

Some few principles expounded by geometricians are, indeed, really *analytical*, and depend on the *principle of contradiction.* Further, Kant says:

> They serve, however, like identical propositions, as links in the chain of method, not as principles- for example, $a = a$, the whole is equal to itself, or $(a+b) > a$, the whole is greater than its part. And yet even these principles themselves, though they derive their validity from pure conceptions, are only admitted in mathematics because they can be presented in intuition. [92]

Kant (1781), in "*The Critic Of Pure Reason: Transcendental Analytic, Book I, Analytic Of Conceptions. Ss 2"* , claims that *through the determination of pure intuition* we obtain *a priori cognitions of mathematical objects*, but only as regards their form as *phenomena.* According to Kant, *all mathematical conceptions,* therefore, are not *per se* cognition, except in so far as we presuppose that there exist *things* which can only be represented conformably to the form of our *pure sensuous intuition.*

[90] *Ibid.*
[91] *Ibid.*
[92] *Ibid.*

Things[93], in *space* and *time* are given only in so far as they are *perceptions* i.e. only by *empirical representation*. Kant insists that the *pure conceptions of the understanding of mathematics,* even when they are applied to intuitions *a priori* , produce *mathematical cognition* only in so far as *these can be applied to empirical intuitions.* Consequently[94], in the *cognition of mathematics,* their application to objects of experience is the only legitimate use of the *categories*.

In *"The Critic of Pure Reason: Appendix",* Kant (1781) elaborates that in the *conceptions of mathematics*, in its pure intuitions, *space has three dimensions,* and *between two points there can be only one straight line,* etc. They[95] would nevertheless have no significance if we were not always able to exhibit their significance in and by means of *phenomena.* It[96] is requisite that an abstract conception be made *sensuous*, that is, that an object corresponding to it in intuition be forth coming, otherwise the conception remains without sense i.e. without meaning. *Mathematics*[97] fulfils this requirement by the *construction of the figure*, which is *a phenomenon evident* to the *senses;* the same science finds support and significance in number; this in its turn finds it in the fingers, or in counters, or in lines and points. The *mathematical*[98] *conception* itself is always produced *a priori*, together with the *synthetical principles* or formulas

[93] Kant, I., 1781, "*The Critic Of Pure Reason: Transcendental Analytic, Book I, Analytic Of Conceptions. Ss 2",* Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>).
[94] *Ibid.*
[95] Kant, I., 1781, *"The Critic Of Pure Reason: Appendix.",* Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>
[96] *Ibid.*
[97] *Ibid.*
[98] *Ibid.*

from such conceptions; but the proper employment of them, and their application to objects, can exist nowhere but in experience, the possibility of which, as regards its form, they contain *a priori*.

Kant in *"The Critic Of Pure Reason: SECTION I. The Discipline of Pure Reason in the Sphere of Dogmatism."*, propounds that, without the aid of experience, the *synthesis in mathematical conception* cannot proceed *a priori* to the intuition which corresponds to the conception. For this reason, none of these conceptions can produce a determinative *synthetical proposition*. They can never present more than a principle of the synthesis of possible empirical intuitions. Kant [99] avows that a *transcendental proposition* is, therefore, a *synthetical cognition* of reason by means of pure conceptions and the *discursive method*. Iit renders possible all *synthetical* unity in empirical cognition, though it cannot present us with any intuition *a priori*. Further, Kant[100] explains that the mathematical conception of *a triangle* we should construct, present *a priori* in intuition and attain to *rational-synthetical cognition.* Kant emphasizes the following:

> But when the *transcendental* conception of reality, or substance, or power is presented to my mind, we find that it does not relate to or indicate either an empirical or pure intuition, but that it indicates merely the synthesis of empirical intuitions, which cannot of course be given *a priori*. [101]

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[99] Kant, I., 1781, *"The Critic Of Pure Reason: SECTION I. The Discipline of Pure Reason in the Sphere of Dogmatism."*, Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>

[100] *Ibid.*

[101] Kant, I., 1781, "The Critic Of Pure Reason: Transcendental Doctrine Of Method; Chapter I. The Discipline Of Pure Reason, Section I. The Discipline Of Pure Reason In The Sphere Of Dogmatism", Translated By J. M. D. Meiklejohn, Retrieved 2003 <*Http://Www.Encarta.Msn. Com/*>

To make clear the notions, Kant sets forth the following:

> Suppose that the conception of a triangle is given to a philosopher and that he is required to discover, by the philosophical method, what relation the sum of its angles bears to a right angle. He has nothing before him but the conception of a figure enclosed within three right lines, and, consequently, with the same number of angles. He may analyze the conception of a right line, of an angle, or of the number three as long as he pleases, but he will not discover any properties not contained in these conceptions. But, if this question is proposed to a geometrician, he at once begins by constructing a triangle. He knows that two right angles are equal to the sum of all the contiguous angles which proceed from one point in a straight line; and he goes on to produce one side of his triangle, thus forming two adjacent angles which are together equal to two right angles. [102]

Mathematical cognition [103] is cognition by means of the construction of conceptions. The construction of a conception is the presentation *a priori* of the intuition which corresponds to the conception. *Mathematics*[104] does not confine itself to the construction of quantities, as in the case of geometry. It occupies itself with pure quantity also, as in the case of *algebra,* where complete abstraction is made of the properties of the object indicated by the conception of quantity. In *algebra*[105], a certain method of notation by signs is adopted, and these indicate the different possible constructions of quantities, the extraction of roots, and so on. *Mathematical cognition*[106] can relate only to *quantity* in which it is to be found in its *form* alone, because the conception of *quantities* only that is capable of being constructed, that is,

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[102] *Ibid*.
[103] Kant, I., 1781, *"The Critic Of Pure Reason: Transcendental Doctrine Of Method, Chapter I, Section I .",* Translated By J. M. D. Meiklejohn, Retrieved 2003*<http://www.encarta.msn. com/>).*
[104] *Ibid.*
[105] *Ibid.*
[106] Kant, I., 1781, "*The Critic Of Pure Reason: SECTION I. The Discipline of Pure Reason in the Sphere of Dogmatism.",* Translated By J. M. D. Meiklejohn, Retrieved 2003*<http://www.encarta.msn. com/>)*

presented *a priori in intuition*; while qualities cannot be given in any other than *an empirical intuition.*

D. Kant on Mathematical Method

Kant's notions of *mathematical method* can be found in "*The Critic Of Pure Reason: Transcendental Doctrine Of Method; Chapter I. The Discipline Of Pure Reason, Section I. The Discipline Of Pure Reason In The Sphere Of Dogmatism".* Kant recites that *mathematical method* is unattended in the sphere of philosophy by the least advantage that *geometry* and *philosophy* are two quite different things, although they go hand in hand in the field of *natural science*, and, consequently, that the procedure of the one can never be imitated by the other. According to Kant[107], the *evidence of mathematics* rests upon *definitions, axioms,* and *demonstrations;* however, *none of these forms can be employed or imitated in philosophy in the sense in which they are understood by mathematicians*. Kant[108] claims that all our *mathematical knowledge* relates to *possible intuitions,* for it is these alone that present objects to the mind. An *a priori* or *non-empirical conception* contains either a *pure intuition* that is it can be constructed; or it contains nothing but the *synthesis of possible intuitions*, which are not given *a priori*. Kant[109] sums up that in this latter case, it may help us to

[107] Kant, I., 1781, "*The Critic Of Pure Reason: Transcendental Doctrine Of Method; Chapter I. The Discipline Of Pure Reason, Section I. The Discipline Of Pure Reason In The Sphere Of Dogmatism",* Translated By J. M. D. Meiklejohn, Retrieved 2003 <*Http://Www.Encarta.Msn. Com/*>*).*
[108] *Ibid.*
[109] *Ibid.*

form *synthetical a priori judgements,* but only in the *discursive method,* by conceptions, not in the intuitive, by means of the construction of conceptions.

On the other hand, Kant[110] explicates that no *synthetical principle* which is based upon conceptions, can ever be *immediately certain*, because we require a *mediating term* to connect the two conceptions of *event* and *cause* that is the condition of *time-determination* in an experience, and we cannot cognize any such principle immediately and from conceptions alone. *Discursive principles* are, accordingly, very different from *intuitive principles* or *axioms*. In his critic, Kant[111] holds that *empirical conception can not be defined, it can only be explained*. In a conception of a certain number of marks or signs, which denote a certain class of sensuous objects, we can never be sure that we do not cogitate under the word which. The science of mathematics alone possesses definitions. According to Kant [112], *philosophical definitions* are merely expositions of given conceptions and are produced by analysis; while, *mathematical definitions* are constructions of conceptions originally formed by the mind itself and are produced by a synthesis.

Further, in a *mathematical definition*[113] the conception is formed; *we cannot have a conception prior to the definition*. Definition gives us the conception. It must form the commencement of every chain of mathematical reasoning. In mathematics[114], *definition* can not be erroneous; it contains only what has been cogitated. However, in

[110] *Ibid*.
[111] Kant, I., 1781, *"The Critic Of Pure Reason: Transcendental Doctrine Of Method, Chapter I, Section I .",* Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>
[112] *Ibid.*
[113] *Ibid.*
[114] *Ibid.*

term of its form, a *mathematical definition* may some*time*s error due to a want of precision. Kant marks that definition: "*Circle is a curved line, every point in which is equally distant from another point called the centre"* is *faulty*, from the fact that the determination indicated by the word curved is *superfluous*. For there ought to be a particular theorem, which may be easily proved from the definition, to the effect that every line, which has all its points at equal distances from another point, must be a curved line (see Figure 22.)- that is, that not even the smallest part of it can be straight.[115]

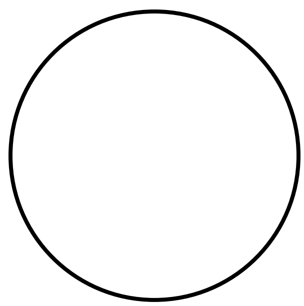

Figure 22: Curve line

Kant (1781) in *"The Critic Of Pure Reason: 1. AXIOMS OF INTUITION, The principle of these is: All Intuitions are Extensive Quantities*", illustrates that *mathematics* have its *axioms* to express the conditions of sensuous intuition *a priori*, under which alone *the schema of a pure conception of external intuition* can exist e.g. *"between two points only one straight line is possible"*, "*two straight lines cannot*

[115] *Ibid.*

enclose a space," etc. These[116] are the *axioms* which properly relate only to quantities as such; but, as regards the quantity of a thing, we have various propositions *synthetical* and *immediately certain* (indemonstrabilia) that they are not the axioms. Kant[117] highlights that the propositions: "*If equals be added to equals, the wholes are equal"; "If equals be taken from equals, the remainders are equal";* are *analytical*, because we are immediately conscious of the identity of the production of the one quantity with the production of the other; whereas *axioms* must be *a priori synthetical* propositions. On the other hand[118], the *self-evident propositions* as to the relation of numbers, are certainly *synthetical* but not universal, like those of *geometry*, and for this reason cannot be called *axioms*, but numerical formulae. Kant[119] proves that $7 + 5 = 12$ is not an analytical proposition; for either in the representation of seven, nor of five, nor of the composition of the two numbers; "*Do I cogitate the number twelve?*" he said.

Although the proposition[120] is *synthetical*, it is nevertheless only a singular proposition. In so far as regard is here had merely to the synthesis of the *homogeneous*, it cannot take place except in one manner, although our use of these numbers is afterwards general. Kant then exemplifies the construction of triangle using three lines as the following:

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[116] Kant, I., 1781, *"The Critic Of Pure Reason: 1. AXIOMS OF INTUITION, The principle of these is: All Intuitions are Extensive Quantities*", Translated By J. M. D. Meiklejohn, Retrieved 2003*<http://www.encarta.msn. com/>).*
[117] *Ibid.*
[118] *Ibid.*
[119] *Ibid.*
[120] *Ibid*.

> The statement: "A triangle can be constructed with three lines, any two of which taken together are greater than the third" is merely the pure function of the productive imagination, which may draw the lines longer or shorter and construct the angles at its pleasure; therefore, such propositions cannot be called as axioms, but numerical formulae[121]

Kant in *"The Critic Of Pure Reason: II. Of Pure Reason as the Seat of Transcendental Illusory Appearance, A. OF REASON IN GENERAL"*, enumerates that *mathematical axioms*[122] are general *a priori* cognitions, and are therefore rightly denominated principles, relatively to the cases which can be subsumed under them. While in "*The Critic Of Pure Reason: SECTION III. Of Opinion, Knowledge, and Belief; CHAPTER III. The Arehitectonic of Pure Reason",* Kant propounds that *mathematics*[123] may possess *axioms,* because it can always connect the *predicates* of an object *a priori*, and without any mediating term, by means of the construction of conceptions in intuition. On the other hand, in *"The Critic Of Pure Reason: CHAPTER IV. The History of Pure Reason; SECTION IV. The Discipline of Pure Reason in Relation to Proofs*" , Kant designates that in *mathematics,* all our conclusions may be drawn immediately from *pure intuition*. Therefore, *mathematical proof* must demonstrate the possibility of arriving, *synthetical*ly and *a priori*, at a certain knowledge of *things*, which was not contained in our conceptions of these

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[121] *Ibid.*

[122] Kant, I., 1781, *"The Critic Of Pure Reason: II. Of Pure Reason as the Seat of Transcendental Illusory Appearance, A. OF REASON IN GENERAL",* Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/>).*

[123] Kant, I., 1781, "*The Critic Of Pure Reason: SECTION III. Of Opinion, Knowledge, and Belief; CHAPTER III. The Arehitectonic of Pure Reason"* Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/>)*

things. All[124] the attempts which have been made to prove *the principle of sufficient reason*, have, according to the universal admission of philosophers, been quite unsuccessful. Before the appearance of *transcendental criticism*, it was considered better to appeal boldly to the *common sense* of mankind, rather than attempt to discover new *dogmatical proofs. Mathematical proof*[125] requires *the presentation of instances of certain concepts*. These instances would not function exactly as particulars, for one would not be entitled to assert anything concerning them which did not follow from the general concept. Kant[126] says that *mathematical method* contains demonstrations because mathematics does not deduce its cognition from conceptions, but from *the construction of conceptions,* that is, from *intuition,* which can be given *a priori* in accordance with conceptions. Ultimately, Kant[127] contends that in *algebraic method,* the correct answer is deduced by *reduction* that *is a kind of construction;* only an *apodeictic proof,* based upon intuition, can be termed a *demonstration.*

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[124] Kant, I., 1781, *"The Critic Of Pure Reason: CHAPTER IV. The History of Pure Reason; SECTION IV. The Discipline of Pure Reason in Relation to Proofs*" Translated By J. M. D. Meiklejohn, Retrieved 2003*<http://www.encarta.msn. com/>)*
[125] Kant in Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York
[126] Kant, I., 1781, "*The Critic Of Pure Reason: Transcendental Doctrine Of Method, Chapter I, Section I .",* Translated By J. M. D. Meiklejohn, Retrieved 2003*<http://www.encarta.msn. com/>).*
[127] *Ibid.*